2021.M29

2021L003A1EL

Coimisiún na Scrúduithe Stáit
State Examinations Commission

Leaving Certificate Examination 2021

Mathematics

Paper 1

Higher Level

Friday 11 June Afternoon 2:00 – 4:30

220 marks

Examination Number

Day and Month of Birth

For example, 3rd February is entered as 0302

Centre Stamp

The 2021 examination papers were adjusted to compensate for disruptions to learning due to COVID-19. This examination paper does not necessarily reflect the same structure and format as the examination papers of past or subsequent years.

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Instructions

There are **two** sections in this examination paper.

Section A	Concepts and Skills	120 marks	6 questions
Section B	Contexts and Applications	100 marks	4 questions

Answer questions as follows:

- any four questions from Section A – Concepts and Skills
- any two questions from Section B – Contexts and Applications.

Write your Examination Number in the box on the front cover.

Write your answers in blue or black pen. You may use pencil in graphs and diagrams only.

This examination booklet will be scanned and your work will be presented to an examiner on screen. Anything that you write outside of the answer areas may not be seen by the examiner.

Write all answers into this booklet. There is space for extra work at the back of the booklet. If you need to use it, label any extra work clearly with the question number and part.

The superintendent will give you a copy of the *Formulae and Tables* booklet. You must return it at the end of the examination. You are not allowed to bring your own copy into the examination.

You will lose marks if your solutions do not include relevant supporting work.

You may lose marks if the appropriate units of measurement are not included, where relevant.

You may lose marks if your answers are not given in simplest form, where relevant.

Write the make and model of your calculator(s) here:

Section A — Concepts and Skills — 120 marks

Answer **any four** questions from this section.

Question 1 **(30 marks)**

(a) $\frac{(4-2i)}{(2+4i)} = 0 + ki$, where $k \in \mathbb{Z}$, and $i^2 = -1$. Find the value of k.

(b) Find $\sqrt{-5+12i}$.
Give both of your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

(c) Use De Moivre's theorem to find the **three** roots of $z^3 = -8$.
Give each of your answers in the form $a + bi$, where $a, b \in \mathbb{R}$, and $i^2 = -1$.

Question 2 **(30 marks)**

(a) Given that $x = -3$ is a solution to $|x + p| = 5$, find the two values of p, where $p \in \mathbb{Z}$.

(b) $(x + 4)$ is a factor of $f(x) = x^3 + qx^2 - 22x + 56$, where $x \in \mathbb{R}$ and $q \in \mathbb{Z}$.

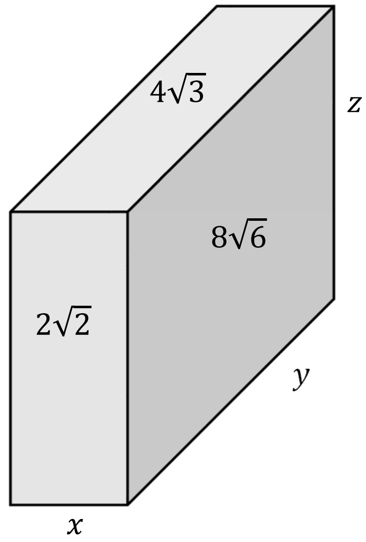
Show that $q = -5$, **and** find the three roots of $f(x)$.

Show:

Roots = $(\quad , \quad , \quad)$

Question 3 **(30 marks)**

The diagram shows a cuboid with dimensions x, y and z cm.
The areas, in cm^2, of three of its faces are also shown.



(a) Find the **volume** of the cuboid in the form $a\sqrt{b}$ cm^3, where $a, b \in \mathbb{N}$.

(b) **(i)** Given that $f(x) = 3x^2 + 8x - 35$, where $x \in \mathbb{R}$, find the two roots of $f(x) = 0$.

(ii) Hence or otherwise, solve the equation $3^{2m+1} = 35 - 8(3^m)$, where $m \in \mathbb{R}$.
Give your answer in the form $m = \log_3 p - q$, where $p, q \in \mathbb{N}$.

Question 4 **(30 marks)**

(a) Prove using induction that $2^{3n-1} + 3$ is divisible by 7 for all $n \in \mathbb{N}$.

(b) $p,\ p+7,\ p+14,\ p+21, \dots \dots$ is an **arithmetic** sequence, where $p \in \mathbb{N}$.

(i) Find the n^{th} term, T_n, in terms of n and p, where $n \in \mathbb{N}$.

(ii) Find the smallest value of p for which 2021 is a term in the sequence.

Question 5 **(30 marks)**

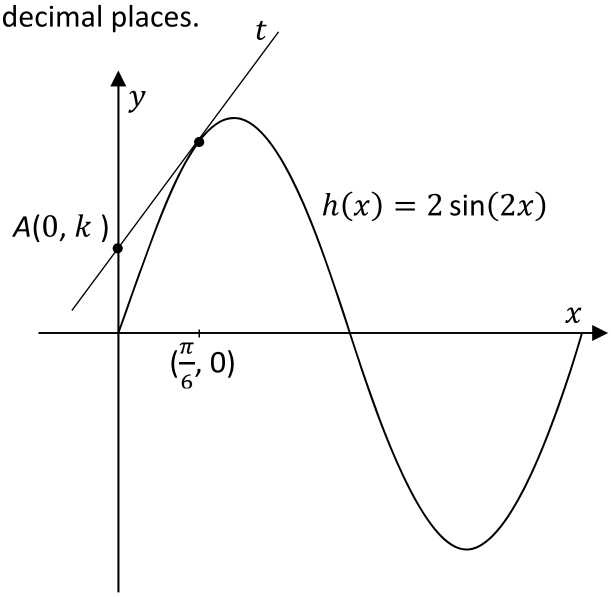
(a) The **derivative** of $f(x) = 2x^3 + 6x^2 - 12x + 3$ can be expressed in the form $f'(x) = a(x + b)^2 + c$, where $a, b, c \in \mathbb{Z}$ and $x \in \mathbb{R}$.

(i) Find the value of a, the value of b, and the value of c.

$a =$ $b =$ $c =$

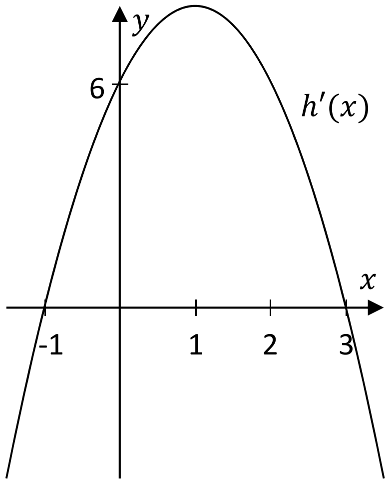
(ii) If $g(x) = 36x + 5$, find the range of values of x for which $f'(x) > g'(x)$.

(b) The diagram below shows t, the tangent line to $h(x) = 2\sin(2x)$, where $0 \leq x \leq \pi$, at the point where $x = \frac{\pi}{6}$.
$A(0, k)$, where $k \in \mathbb{R}$, is the point where t cuts the y-axis.
Find the value of k correct to two decimal places.



Question 6 **(30 marks)**

The diagram below shows the graph of $h'(x)$ the derivative of a cubic function $h(x)$.



(a) Show that $h'(x) = -2x^2 + 4x + 6$.

(b) Use $h'(x)$ to find the maximum positive value of the **slope** of a tangent to $h(x)$.

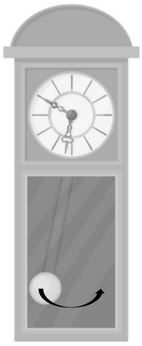
(c) The graph of $h(x)$ passes through the point $(0, -2)$.
Find the equation of $h(x)$.

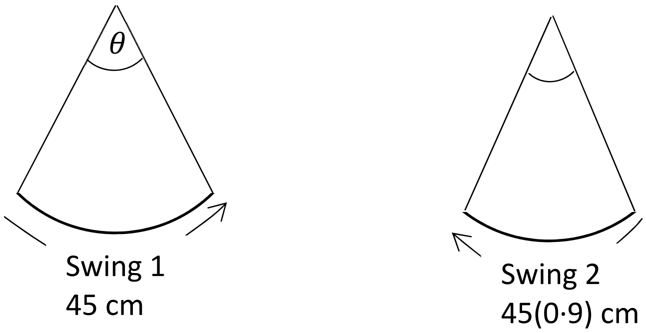
Section B	Contexts and Applications	100 marks

Answer **any two** questions from this section.

Question 7 **(50 marks)**

The tip of the pendulum of a grandfather clock swings initially through an arc length of 45 cm.
On each successive swing the length of the arc is 90% of the previous length.





(a) **(i)** Complete the table below by filling in the missing lengths.

Swing	1	2	3	4	5
Length of Arc (cm)	45		$\frac{729}{20}$		

(ii) $T_n = 45(0{\cdot}9)^{n-1}$ is the arc length of swing n.
Find the arc length of swing 25, correct to 1 decimal place.

(iii) Find the **total** distance travelled by the tip of the pendulum when it has completed swing 40. Give your answer, in cm, correct to the nearest whole number.

(iv) Swing p is the first swing which has an arc length of less than 2 cm. Find the value of p.

This question continues on the next page.

(b) **(i)** If the length of the pendulum is 1 m, show that the angle, θ, of swing 1 of the pendulum is $26°$, correct to the nearest degree.

(ii) Hence, find the total accumulated angle that the pendulum swings through (i.e. the sum of all the angles it swings through until it stops swinging).
Give your answer correct to the nearest degree.

(iii) Hence, or otherwise, find the **total distance** travelled by the tip of the pendulum when it has moved through half of the total accumulated angle.
Give your answer, in cm, correct to the nearest integer.

Question 8 **(50 marks)**

(a) The table in **Part (a)(ii)** below shows some of the values of the function:

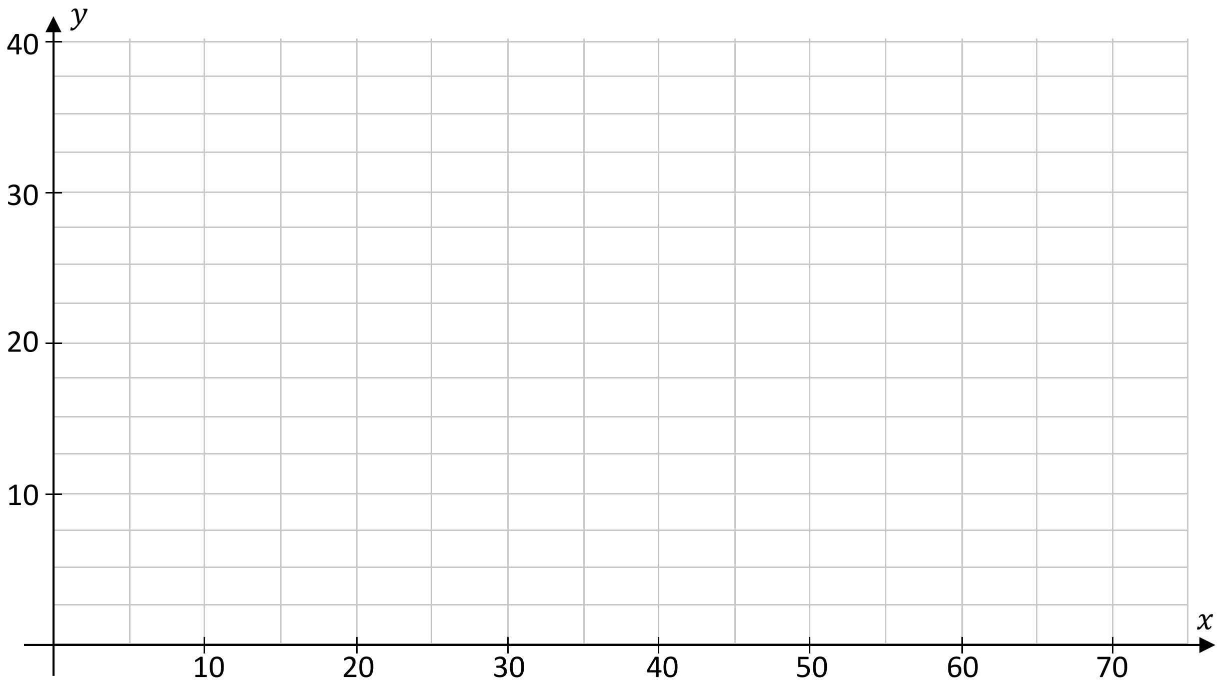
$$h(x) = 0{\cdot}001x^3 - 0{\cdot}12x^2 + px + 5, \quad x \in \mathbb{R},$$

in the domain $0 \leq x \leq 75$.

(i) Use $h(10) = 30$ to show that $p = 3{\cdot}6$

(ii) Complete the table below and hence draw the graph of $h(x)$ in the domain $0 \leq x \leq 75$ on the grid below.

x	0	10	20	30	40	50	60	70	75
$h(x)$		30			21		5		21·875



(b) The function $h(x)$ can be used to model the height above level ground (in metres) of a section of the path followed by a rollercoaster track, where x is the horizontal distance from a fixed point.

(i) Find $h'(x)$, the derivative of $h(x)$.

(ii) Show that this section of the track reaches its maximum height above level ground when $x = 20$.

This question continues on the next page.

(iii) Find, using calculus, the height above ground, in metres, at the instant the track passes through an inflection point. The function $h(x) = 0{\cdot}001x^3 - 0{\cdot}12x^2 + 3{\cdot}6x + 5$.

(c) Use the function $h(x) = 0{\cdot}001x^3 - 0{\cdot}12x^2 + 3{\cdot}6x + 5$, $x \in \mathbb{R}$, to find the average height of this section of the track above level ground, from $x = 0$ to $x = 75$.
Give your answer in metres correct to 2 decimal places.

Question 9 **(50 marks)**

(a) A cup of coffee is freshly brewed to 95° C.
The temperature, T, in degrees centigrade, of the coffee as it cools is given by the formula

$$T(t) = Ae^{-0\cdot081t} + 20$$

where A is constant and t is time measured in minutes from when the coffee was brewed.

(i) Show that $A = 75$.

(ii) Explain what the value 20 in the formula represents in the context of the coffee cooling.

(iii) Find the **decrease** in the temperature of the coffee 10 minutes after brewing.
Give your answer correct to the nearest whole number.

This question continues on the next page.

(b) $T(t) = 75e^{-0\cdot081t} + 20$ gives the temperature of the coffee at time t.
If the ideal temperature to drink coffee is 82°C, find the time, to the nearest second, that it takes for the coffee to reach this temperature.

(c) Find, to the nearest °C, the temperature the coffee has reached when $T'(t) = -4\cdot05$, where $T'(t)$ is the rate at which the coffee is cooling, in °C per minute.

(d) A sugar cube is put into the coffee.
The sugar keeps its cube shape as it dissolves.
As the sugar cube dissolves, its volume decreases at the constant rate of $\frac{1}{20}$ cm^3/sec.
Let $x(t)$ be the sidelength of the cube at time t.
Find the rate of change of $x(t)$ when the volume of the cube reaches $\frac{1}{64}$ cm^3.

Question 10 **(50 marks)**

(a) Water is flowing into and being removed from a tank.
The volume of water in the tank is measured on a daily basis, starting at day 0.
The volume of water, in litres, in the tank can be modelled by the formula

$$V(t) = 60 + 41t - 3t^2, \text{ while } V(t) \geq 0,$$

where $t \in \mathbb{R}$ is the time in days, starting at day 0.
Use the function $V(t)$ to answer the following 4 questions.

(i) Find the value of t when the tank empties.

(ii) Find the rate at which the volume of water in the tank is changing when $t = 5$.

(iii) Find the value of t when the volume of water in the tank is a maximum.

(iv) Find the maximum volume of the water in the tank, correct to the nearest litre.

This question continues on the next page.

(b) Trees of the same age, size and type are growing at the same rate in a forest.
It is possible to measure the radius of the trunk of a tree at the end of each growing season.
The **increase** in the radius of the trunks of these trees **each year**, in cm,
during a particular 10 year period can be modelled by the formula:

$$I(t) = 1{\cdot}5 + \sin\frac{\pi t}{5},$$

where $1 \leq t \leq 10$ and $t \in \mathbb{N}$ is measured in years.

Therefore
$$r(1) = r(0) + I(1),$$
$$r(2) = r(1) + I(2),\ \ldots,$$
$$r(t+1) = r(t) + I(t+1)$$

where $r(t)$ is the radius of the trunk of the tree after t years
and $r(0)$ is the radius of the trunk of a tree at the beginning of year 1.

(i) In order for the model to be reasonable it must satisfy a number of conditions.
One condition is written below:

- *The radius of the trees is increasing year on year*.

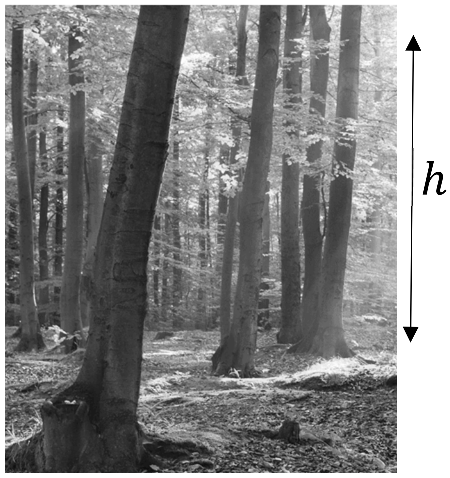
Show that $r(t)$ satisfies this condition.

(ii) Show that $I(6) < I(5)$ and explain what this means in the context of the growth of a tree.

Show:

Explanation:

(iii) Two identical trees are growing in the forest.
At the beginning of year 1, a tree was cut down and a section of its trunk, in the shape of a cylinder of radius 10 cm, standard length h cm, and volume V_1 cm^3 was sent to a sawmill.
Given that $r(0) = 10$, the formula
$r(t+1) = r(t) + I(t+1)$, where $t \in \mathbb{N}$,
can be used to find the radius of the second tree for subsequent values of t.
Find $r(2)$, the radius of the second tree at $t = 2$.
Give your answer in the form $a + \sin\frac{b\pi}{5} + \sin\frac{c\pi}{5}$,
where $a, b,$ and $c \in \mathbb{N}$.



(iv) At $t = 10$ the second tree was also cut down.
A section of its trunk, in the shape of a cylinder, of radius $r(10)$ cm, standard length h cm, and volume V_2 cm^3 was also sent to a sawmill.
If $V_2 = kV_1$, where $k \in \mathbb{R}$, find the value of k.

Page for extra work.
Label any extra work clearly with the question number and part.

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Leaving Certificate – Higher Level

Mathematics Paper 1

Friday 11 June

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